CONDUCTORS AND NEWFORMS FOR NON-SUPERCUSPIDAL REPRESENTATIONS OF UNRAMIFIED $\mathrm{U}(2,1)$

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Abstract. Let G be the unramified unitary group in three variables defined over a p-adic field with $p \neq 2$. The conductors and newforms for representations of G are defined by using a certain family of open compact subgroups of G. In this paper, we determine the conductors of non-supercuspidal representations of G, and give explicit newforms in induced representations for non-supercuspidal generic representations.

Introduction

Local newforms play an important role in the theory of automorphic representation. We recall a result of Casselman [3] for $\mathrm{GL}(2)$. Let F be a non-archimedean local field of characteristic zero and $\mathfrak{o}_F$ its ring of integers with maximal ideal $\mathfrak{p}_F$. For each non-negative integer n, we define an open compact subgroup $\Gamma_0(\mathfrak{p}_F^n)$ of $\mathrm{GL}_2(F)$ by

$$\Gamma_0(\mathfrak{p}_F^n) = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F^n & 1+\mathfrak{p}_F^n \end{pmatrix}^\times .$$

Let (π, V) be an irreducible admissible representation of $\mathrm{GL}_2(F)$. We set $V(n) = \{v \in V \,|\, \pi(k)v = v,\ k \in \Gamma_0(\mathfrak{p}_F^n)\}$. Then the following holds:

Theorem 0.1 ([3]). *Suppose that an irreducible admissible representation (π, V) of $\mathrm{GL}_2(F)$ is generic.*

(1) There exists a non-negative integer n such that $V(n)$ is not zero.

(2) Put $c(\pi) = \min\{n \geq 0 \,|\, V(n) \neq \{0\}\}$. Then the space $V(c(\pi))$ is one-dimensional.

(3) For any integer $n \geq c(\pi)$,

$$\dim V(n) = n - c(\pi) + 1.$$

(4) The ε-factor of π is a constant multiple of $q_F^{-c(\pi)s}$ under suitable normalization, where q_F is the cardinality of the residue field of F.

We call the integer $c(\pi)$ *the conductor of* π and elements in $V(c(\pi))$ *newforms for* π. In the proof of Theorem 0.1, Casselman showed implicitly that newforms are test vectors for the appropriate Whittaker functional. This property is important in the theory of zeta integrals. After Casselman [3], similar results are obtained by Jacquet, Piatetski-Shapiro and Shalika [6] and Reeder [13] for $\mathrm{GL}_n(F)$ and by Roberts and Schmidt [14] for $\mathrm{PGSp}(4)$. For unitary groups, there is a result by Lansky and Raguram [8]. They computed the dimensions of the spaces of vectors fixed by certain open compact subgroups of unramified $\mathrm{U}(1,1)$. However no comparison between conductors and exponents of ε-factors has been studied for this group.

The author [11] introduced the notion of newforms for the unramified unitary group in three variables defined over F. We assume that F is in addition of odd residual characteristic. Let E be the unramified quadratic extension over F. The group unramified $\mathrm{U}(2,1)$ is realized as $G = \{g \in \mathrm{GL}_3(E) \mid {}^t\overline{g}Jg = J\}$, where $^-$ is the non-trivial element in $\mathrm{Gal}(E/F)$ and J is

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$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Newforms for G is defined by a family of open compact subgroups $\{K_n\}_{n\geq 0}$, where

$$K_n = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^n & 1+\mathfrak{p}_E^n & \mathfrak{o}_E \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{o}_E \end{pmatrix} \cap G.$$

For a smooth representation (π, V) of G, we denote by $V(n)$ the space of K_n-fixed vectors in V. We say that a smooth representation (π, V) admits a newform if $V(n)$ is not zero for some $n \geq 0$. In [11], the author showed that every irreducible generic representation of G admits a newform. The integer $N_\pi = \min\{n \geq 0 \,|\, V(n) \neq \{0\}\}$ is called the conductor of π. We call $V(N_\pi)$ the space of newforms for π and $V(n)$ that of oldforms when $n > N_\pi$. The main theorem of [11] is the multiplicity one property of newforms, that is, for each irreducible admissible representation π of G admitting a newform, the space of newforms for π is one-dimensional.

One of aims of this paper is to show that newforms for G are test vectors for the appropriate Whittaker functional. This property is important for the application to the theory of zeta integral. In [10], the author applied newform theory for G to Rankin-Selberg type zeta integrals of Gelbart, Piatetski-Shapiro [5] and Baruch [1], and proved that zeta integrals of newforms for generic supercuspidal representations agree with their L-factors. This property means that zeta integrals of newforms do not vanish.

To show that newforms are test vectors for the Whittaker functional, we determine the newforms for each non-supercuspidal generic representation explicitly. This property was already proved in [11] for a certain class of representations of G, which contains the generic supercuspidal representations. Therefore we need to consider only non-supercuspidal representations. Similar results are obtained by Lansky and Raguram for unramified $\mathrm{U}(1,1)$ and for $\mathrm{SL}(2)$([8] and [9]), and by Roberts and Schmidt for $\mathrm{PGSp}(4)$ ([14]).

Firstly, we determine conductors and oldforms for the parabolically induced representations from the Borel subgroup B of G. A parabolically induced representation (π, V) of G is induced from a quasi-character μ_1 of $E^\times$ and a character μ_2 of the norm-one subgroup E^1 of $E^\times$. By the theory of Bernstein and Zelevinsky, to obtain a basis for $V(n)$, we only need to determine the elements g in $B\backslash G/K_n$ such that $\mu_1 \otimes \mu_2$ is trivial on $B \cap gK_ng^{-1}$. It will turn out that the conductor of $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$ equals to $2c(\mu_1) + c(\mu_2)$, where $c(\mu_i)$ is the conductor of μ_i (see Theorem 2.4).

Secondly, we determine the conductors of irreducible subquotients of reducible parabolically induced representations according to the classification by Keys [7]. Here the level raising operator θ' on the space of newforms plays an important role. If we find a vector which does not vanish under θ', then this vector should belong to the generic constituent of the parabolically induced representation. The method in this part is rather technical.

Thirdly, we determine newforms for generic non-supercuspidal representations explicitly. Every irreducible non-supercuspidal representation π of G can be embedded into $\mathrm{Ind}_B^G \mu$, for some quasi-character μ of the diagonal torus T. We realize the newform for π as a function in $\mathrm{Ind}_B^G \mu$ explicitly. This is easy if π is not the Steinberg representation. The space of newform for the Steinberg representation is characterized as the kernel of the level lowering operator on oldforms for $\mathrm{Ind}_B^G \mu$.

Finally, we prove that newforms are test vectors for the appropriate Whittaker functional. Here the level raising operator θ' on the space of newforms plays an important role again. Using the explicit newforms, we show that the injectivity of θ'. Then this property follows automatically (see Lemma 1.7). As a corollary, we obtain the following dimension formula of oldforms. Similar to the case of $\mathrm{GL}(n)$ and $\mathrm{GSp}(4)$, the growth of dimensions of oldforms for generic representations π of G is independent of the choice of π.

Theorem 0.2. *Let (π, V) be an irreducible generic representation of G. For $n \geq N_\pi$, we have*

$$\dim V(n) = \left\lfloor \frac{n - N_\pi}{2} \right\rfloor + 1.$$

We hope that results in this paper are useful for the theory of zeta integral, especially Rankin-Selberg type one of Gelbart and Piatetski-Shapiro.

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1. Preliminaries

In subsection 1.1, we fix notation for the unramified unitary group in three variables, which is used in this paper. In subsection 1.2, we recall from [11] the definition and some properties of newforms for unramified $\mathrm{U}(2,1)$. As in [11], the level raising operator θ' on the space of newforms plays an important role. We will prepare Lemma 1.7, which is a main tool in our investigation.

1.1. Notation.

Let F be a non-archimedean local field of characteristic zero, $\mathfrak{o}_F$ its ring of integers, $\mathfrak{p}_F = \varpi_F \mathfrak{o}_F$ the maximal ideal in $\mathfrak{o}_F$ and $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue field of F. We write $q = q_F$ for the cardinality of k_F. Let $|\cdot|_F$ be the absolute value of F normalized so that $|\varpi_F|_F = q_F^{-1}$. We use the analogous notation for any non-archimedean local field. Throughout this paper, we assume that F is of odd residual characteristic.

Let $E = F[\sqrt{\epsilon}]$ be the quadratic unramified extension over F, where ϵ is a non-square element in $\mathfrak{o}_F^\times$. Then ϖ_F is a uniformizer of E and we abbreviate $\varpi = \varpi_F$. We realize the F-points of the unramified unitary group in three variables defined over F as $G = \{g \in \mathrm{GL}_3(E) \mid {}^t\overline{g}Jg = J\}$, where $^-$ is the non-trivial element in $\mathrm{Gal}(E/F)$ and

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let B be the Borel subgroup of G consisting of the upper triangular elements in G, T the Levi subgroup of B and U the unipotent radical of B. We write $\hat{U}$ for the opposite of U. Then we have

$$U = \left\{ u(x,y) = \begin{pmatrix} 1 & x & y \\ 0 & 1 & -\overline{x} \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y \in E,\ y + \overline{y} + x\overline{x} = 0 \right\}$$

and

$$\hat{U} = \left\{ \hat{u}(x,y) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & -\overline{x} & 1 \end{pmatrix} \,\middle|\, x, y \in E,\ y + \overline{y} + x\overline{x} = 0 \right\}.$$

We fix a non-trivial additive character ψ_E of E with conductor $\mathfrak{o}_E$ and define a character ψ of U by

$$\psi(u(x,y)) = \psi_E(x), \text{ for } u(x,y) \in U.$$

We say that a smooth representation π of G is *generic* if $\mathrm{Hom}_U(\pi, \psi) \neq \{0\}$. For an irreducible admissible representation π of G, it is well-known that

$$\dim \mathrm{Hom}_U(\pi, \psi) \leq 1.$$

If (π, V) is an irreducible generic representation of G, then by Frobenius reciprocity, we have

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \psi) \simeq \mathrm{Hom}_U(\pi, \psi) \simeq \mathbf{C}.$$

So there exists a unique embedding of π into $\mathrm{Ind}_U^G\psi$ up to scalar. The image $\mathcal{W}(\pi,\psi)$ of V is called *the Whittaker model of* π. By a non-zero functional $l \in \mathrm{Hom}_U(\pi,\psi)$, which is called *the Whittaker functional*, we define *the Whittaker function* $W_v \in \mathcal{W}(\pi,\psi)$ associated to $v \in V$ by

$$W_v(g) = l(\pi(g)v),\ \ g \in G.$$

We set

$$T_H = \left\{ t(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \,\middle|\, a \in E^\times \right\}.$$

Then the group T_H is isomorphic to $E^\times$. Let (π, V) be an irreducible generic representation of G. For each $v \in V$, we can regard the restriction $W_v|_{T_H}$ of W_v to T_H as a locally constant function on $E^\times$. Along the lines of the Kirillov theory for GL(2), we see that there exists an integer n such that $\mathrm{supp}\, W_v|_{T_H} \subset \mathfrak{p}_E^{-n}$. Moreover, if v lies in $\langle \pi(u)w - w \mid u \in U,\ w \in V\rangle$, then $W_v|_{T_H}$ is a compactly supported function on $E^\times$.

The conductor of a quasi-character μ_1 of $E^\times$ is defined by

$$c(\mu_1) = \min\{n \geq 0 \mid \mu_1|_{(1+\mathfrak{p}_E^n)\cap \mathfrak{o}_E^\times} = 1\}.$$

We say that μ_1 is *unramified* if $c(\mu_1) = 0$. We set open compact subgroups of the norm-one subgroup E^1 of $E^\times$ as

$$E_0^1 = E^1,\ E_n^1 = E^1 \cap (1 + \mathfrak{p}_E^n),\ \text{for } n \geq 1.$$

We define the conductor of a character μ_2 of E^1 by

$$c(\mu_2) = \min\{n \geq 0 \mid \mu_2|_{E_n^1} = 1\}.$$

There exists an isomorphism between E^1 and the center Z of G given by

$$\iota : E^1 \simeq Z; \lambda \mapsto \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}.$$

If a smooth representation π of G admits the central character ω_π, then we define *the conductor of* ω_π by

$$n_\pi = \min\{n \geq 0 \mid \omega_\pi|_{Z_n} = 1\},$$

where $Z_n = \iota(E_n^1)$ for $n \geq 0$.

We shall prepare the following lemma on the structure of E^1.

Lemma 1.1. *Suppose that a subgroup H of E^1 contains the set $\{(1-a\sqrt{\epsilon})(1+a\sqrt{\epsilon})^{-1} \mid a \in \mathfrak{o}_F\}$. Then we have $H = E^1$.*

Proof. By [12] Theorem 2.13 (c), the group E_1^1 coincides with $\{(1-a\sqrt{\epsilon})(1+a\sqrt{\epsilon})^{-1} \mid a \in \mathfrak{p}_F\}$. So H contains E_1^1 by assumption. The quotient E^1/E_1^1 is isomorphic to the norm-one subgroup k_E^1 of $k_E^\times$, and hence it is a cyclic group of order $q+1$. We claim that for $a, b \in \mathfrak{o}_F$, $(1-a\sqrt{\epsilon})(1+a\sqrt{\epsilon})^{-1} \equiv (1-b\sqrt{\epsilon})(1+b\sqrt{\epsilon})^{-1} \pmod{E_1^1}$ implies $a \equiv b \pmod{\mathfrak{p}_F}$. Then the group H/E_1^1 contains at least q-elements. Thus we get $H/E_1^1 = E^1/E_1^1$, whence $H = E^1$.

We shall prove the claim. Suppose that two elements a, b in $\mathfrak{o}_F$ satisfy $(1-a\sqrt{\epsilon})(1+a\sqrt{\epsilon})^{-1} \equiv (1-b\sqrt{\epsilon})(1+b\sqrt{\epsilon})^{-1} \pmod{E_1^1}$. Then we have $(1-a\sqrt{\epsilon})(1+a\sqrt{\epsilon})^{-1} - (1-b\sqrt{\epsilon})(1+b\sqrt{\epsilon})^{-1} \in \mathfrak{p}_E$. Since $(1+a\sqrt{\epsilon})(1+b\sqrt{\epsilon})$ lies in $\mathfrak{o}_E^\times$, we obtain $(1-a\sqrt{\epsilon})(1+b\sqrt{\epsilon}) - (1-b\sqrt{\epsilon})(1+a\sqrt{\epsilon}) = 2(b-a)\sqrt{\epsilon} \in \mathfrak{p}_E$. This means $b - a \in \mathfrak{p}_F$ since we are assuming that F is of odd residual characteristic. $\square$

1.2. **Newforms.** For each non-negative integer n, we define an open compact subgroup K_n of G by

$$K_n = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^n & 1+\mathfrak{p}_E^n & \mathfrak{o}_E \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{o}_E \end{pmatrix} \cap G.$$

We put

$$t_n = \begin{pmatrix} & & \varpi^{-n} \\ & 1 & \\ \varpi^n & & \end{pmatrix} \in K_n.$$

For a smooth representation (π, V) of G, we set

$$V(n) = \{v \in V \mid \pi(k)v = v,\ k \in K_n\},\ n \geq 0.$$

Definition 1.2 ([11] Definition 2.6)**.** Suppose that a smooth representation (π, V) of G has a non-zero K_n-fixed vector, for some $n \geq 0$. We define *the conductor of* π by $N_\pi = \min\{n \geq 0 \,|\, V(n) \neq \{0\}\}$. We call $V(N_\pi)$ *the space of newforms for* π and $V(n)$ *that of oldforms*, for $n > N_\pi$.

We say that a smooth representation π admits a newform if π has a non-zero K_n-fixed vector, for some $n \geq 0$.

Theorem 1.3 ([11] Theorems 2.8, 5.6)**.** *(i) Every irreducible generic representation of G admits a newform.*

(ii) If an irreducible admissible representation (π, V) of G admits a newform, then $V(N_\pi)$ is one-dimensional.

Remark 1.4. Suppose that a smooth representation π of G admits the central character ω_π. If π has a non-zero K_n-fixed vector, then ω_π is trivial on $Z_n = Z \cap K_n$. So we get

$$N_\pi \geq n_\pi.$$

In [11], two level raising operators, which are inspired by those in [14], played an important role to investigate K_n-fixed vectors. The first one $\eta : V(n) \to V(n+2)$ is given by

$$\eta v = \pi(\zeta^{-1})v,\ v \in V(n),$$

where

$$\zeta = \begin{pmatrix} \varpi & & \\ & 1 & \\ & & \varpi^{-1} \end{pmatrix}.$$

The second level raising operator $\theta' : V(n) \to V(n+1)$ is defined by

$$\theta' v = \frac{1}{\mathrm{vol}(K_{n+1} \cap K_n)} \int_{K_{n+1}} \pi(k)v dk,\ v \in V(n).$$

By [11] Proposition 3.3, we have

$$\theta' v = \eta v + \sum_{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} v,\ v \in V(n). \tag{1.5}$$

If newforms for π are test vectors for the Whittaker functional, we have the following dimension formula of oldforms:

Proposition 1.6 ([11] Theorem 5.8). *Let (π, V) be an irreducible generic representation of G. Suppose that $W_v(1) \neq 0$ for all non-zero elements v in $V(N_\pi)$. Then, for $n \geq N_\pi$, the set $\{\theta'^i \eta^j v \mid i + 2j + N_\pi = n\}$ constitutes a basis for $V(n)$. In particular,*

$$\dim V(n) = \left\lfloor \frac{n - N_\pi}{2} \right\rfloor + 1.$$

It follows from [11] Theorem 4.12 that the assumption of Proposition 1.6 holds for irreducible generic representations of G which satisfy $N_\pi \geq 2$ and $N_\pi > n_\pi$. One of the aims of this paper is to prove the above dimension formula of oldforms for all generic representations of G. The following lemma gives a criterion for newforms to be test vectors for the Whittaker functional.

Lemma 1.7. *Let (π, V) be an irreducible admissible representation of G which admits a newform. Suppose that $N_\pi \geq 1$ and the level raising operator $\theta' : V(N_\pi) \to V(N_\pi + 1)$ is injective. Then*

(i) π is generic;

(ii) For all non-zero elements v in $V(N_\pi)$, we have $W_v(1) \neq 0$.

Proof. (i) By assumption, the space $V(N_\pi + 1)$ is not zero. Thus it follows from [11] Theorem 5.10 (ii) that π must be generic.

(ii) Due to [11] Corollary 4.6, for $v \in V(n)$, the function $W_v|_{T_H}$ is $\mathfrak{o}_E^\times$-invariant and its support is contained in $\mathfrak{o}_E$. Let v be an element in $V(N_\pi)$ such that $W_v(1) = 0$. By (1.5), we have

$$W_{\theta' v}(t(a)) = W_v(t(a\varpi^{-1})) + qW_v(t(a)),$$

for $a \in E^\times$, so that $W_{\theta' v}(1) = W_v(t(\varpi^{-1})) + qW_v(1) = 0$. This implies that the support of $W_{\theta' v}|_{T_H}$ is contained in $\mathfrak{p}_E$. It follows from [11] Lemma 4.9 that $\theta' v$ lies in $\eta V(N_\pi - 1) = \{0\}$. So we get $v = 0$ by the injectivity of θ'. Now the proof is complete. $\square$

2. Conductors of parabolically induced representations

In this section, we determine the conductors and oldforms for the parabolically induced representations of G. Here we do not assume that they are irreducible. We use the following notation for parabolically induced representations of G. Given a quasi-character μ_1 of $E^\times$ and a character μ_2 of E^1, we define a quasi-character $\mu = \mu_1 \otimes \mu_2$ of T by

$$\mu \begin{pmatrix} a & & \\ & b & \\ & & \overline{a}^{-1} \end{pmatrix} = \mu_1(a)\mu_2(b), \text{ for } a \in E^\times \text{ and } b \in E^1.$$

We regard μ as a quasi-character of B by extending trivially on U. Let $\pi = \mathrm{Ind}_B^G(\mu)$ be the normalized parabolic induction. Then the space V of π is that of locally constant functions $f : G \to \mathbf{C}$ which satisfy

$$f(bg) = \delta_B(b)^{1/2}\mu(b)f(g), \text{ for } b \in B,\ g \in G,$$

where δ_B is the modulus character of B. We note that

$$\delta_B \begin{pmatrix} a & & \\ & b & \\ & & \overline{a}^{-1} \end{pmatrix} = |a|_E^2, \text{ for } a \in E^\times,\ b \in E^1.$$

For a non-negative integer n, it follows from [2] (2.24) that $V(n)$ is spanned by the functions supported on BgK_n, where g runs over the elements in $B\backslash G/K_n$ such that μ is trivial on $B \cap gK_ng^{-1}$. For any set S, we denote by $\mathrm{Card}\, S$ the cardinality of S. Thus we have

$$\dim V(n) = \mathrm{Card}\{g \in B\backslash G/K_n \mid \mu|_{B\cap gK_ng^{-1}} = 1\}.$$

We shall give a complete set of representatives for $B\backslash G/K_n$. For any integer i, we set

$$\gamma_i = \begin{pmatrix} 1 & & \\ \varpi^i & 1 & \\ -\varpi^{2i}/2 & -\varpi^i & 1 \end{pmatrix} \in G.$$

Lemma 2.1. *For a non-negative integer n, a complete set of representatives for $B\backslash G/K_n$ is given by $\lfloor \frac{n}{2} \rfloor + 1$ elements*

$$\gamma_i, \ \left\lceil \frac{n}{2} \right\rceil \leq i \leq n.$$

Proof. Firstly, we claim that $\mathrm{Card}(B\backslash G/K_n) = \lfloor \frac{n}{2} \rfloor + 1$. It follows [7] that there exists an unramified quasi-character μ_1 of $E^\times$ such that $\pi = \mathrm{Ind}_B^G \mu_1 \otimes 1$ is irreducible. Clearly, $\mu_1 \otimes 1$ is trivial on $B \cap gK_n g^{-1}$ for all $g \in G$. This means $\mathrm{Card}(B\backslash G/K_n) = \dim V(n)$. The representation $\pi = \mathrm{Ind}_B^G \mu_1 \otimes 1$ is generic and unramified, that is, π has conductor zero. Then [4] Theorem 5.4 says that $W_v(1) \neq 0$ for every non-zero element v in $V(0)$. So by Proposition 1.6, we have

$$\mathrm{Card}(B\backslash G/K_n) = \dim V(n) = \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

as required.

Secondly, we show that every element in G lies in $B\gamma_i K_n$, for some $\lceil \frac{n}{2} \rceil \leq i \leq n$. If $n = 0$, the assertion follows from the Iwasawa decomposition $G = BK_0$. The group $K_0/(1 + M_3(\mathfrak{p}_E)) \cap G$ is isomorphic to $\mathrm{U}(2,1)(k_E/k_F)$. Using the Bruhat decomposition of $\mathrm{U}(2,1)(k_E/k_F)$, we get $K_0 = (B \cap K_0)W(K_0 \cap K_1)$, where $W = \{1, t_0\}$. Thus we have $G = BK_0 = BW(K_0 \cap K_1) = BK_1$ since $t_0 \in Bt_1 \subset BK_1$. This completes the proof for $n = 1$. Suppose that $n \geq 2$. Set $I = Z(K_0 \cap K_1)$. Then I is the standard Iwahori subgroup of G and have the Iwahori decomposition $I = (I \cap \hat{U})(I \cap T)(I \cap U)$. So we get $G = BWI = B(I \cap \hat{U}) \cup Bt_0(I \cap U)$, and hence $G = G\zeta^j = B(I \cap \hat{U})\zeta^j \cup Bt_0(I \cap U)\zeta^j$, where $j = \lfloor \frac{n}{2} \rfloor$. Set

$$\hat{U}(\mathfrak{p}_E^{\lceil \frac{n}{2} \rceil}) = \begin{pmatrix} 1 & & \\ \mathfrak{p}_E^{\lceil \frac{n}{2} \rceil} & 1 & \\ \mathfrak{p}_E^n & \mathfrak{p}_E^{\lceil \frac{n}{2} \rceil} & 1 \end{pmatrix} \cap G.$$

We shall claim that $G = B\hat{U}(\mathfrak{p}_E^{\lceil \frac{n}{2} \rceil})K_n$. Clearly, $B(I \cap \hat{U})\zeta^j = B\zeta^{-j}(I \cap \hat{U})\zeta^j$ is contained in $B\hat{U}(\mathfrak{p}_E^{\lceil \frac{n}{2} \rceil})K_n$. Moreover, we have $Bt_0(I \cap U)\zeta^j \subset Bt_0(I \cap U)\zeta^j K_n = B\zeta^{j-n}t_0(I \cap U)\zeta^j t_n K_n = Bt_{n-j}(I \cap U)t_{n-j}K_n \subset B\hat{U}(\mathfrak{p}_E^{\lceil \frac{n}{2} \rceil})K_n$. So we obtain $G = B\hat{U}(\mathfrak{p}_E^{\lceil \frac{n}{2} \rceil})K_n$, as required. Every element in $\hat{U}(\mathfrak{p}_E^{\lceil \frac{n}{2} \rceil})$ can be written as $\hat{u}(y, a\sqrt{\epsilon} - y\overline{y}/2)$, where $y \in \mathfrak{p}_E^{\lceil \frac{n}{2} \rceil}$ and $a \in \mathfrak{p}_F^n$. Since $\hat{u}(y, a\sqrt{\epsilon} - y\overline{y}/2) = \hat{u}(y, -y\overline{y}/2)\hat{u}(0, a\sqrt{\epsilon})$ and $\hat{u}(0, a\sqrt{\epsilon}) \in K_n$, any element in G belongs to $B\hat{u}(y, -y\overline{y}/2)K_n$, for some $y \in \mathfrak{p}_E^{\lceil \frac{n}{2} \rceil}$. If y lies in $\mathfrak{p}_E^n$, then the element $\hat{u}(y, -y\overline{y}/2)$ belongs to K_n. Hence we have $B\hat{u}(y, -y\overline{y}/2)K_n = BK_n = B\gamma_n K_n$. Suppose that y lies in $\mathfrak{p}_E^{\lceil \frac{n}{2} \rceil} \backslash \mathfrak{p}_E^n$. Put $i = \nu_E(y)$, where ν_E is the valuation on E normalized so that $\nu_E(\varpi) = 1$. Then we have $B\hat{u}(y, -y\overline{y}/2)K_n = B\gamma_i K_n$ since B and K_n contain $T_H \cap K_0$. This completes the proof. □

The following lemma determines the condition on i such that μ is trivial on $B \cap \gamma_i K_n \gamma_i^{-1}$.

Lemma 2.2. *Let i be an integer such that $\lceil \frac{n}{2} \rceil \leq i \leq n$. Then μ is trivial on $B \cap \gamma_i K_n \gamma_i^{-1}$ if and only if*

$$c(\mu_2) \leq 2i - n \text{ and } c(\mu_1) \leq n - i.$$

Proof. We can write an element g in $B = TU$ as

$$g = \begin{pmatrix} a & & \\ & b & \\ & & \overline{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & -\overline{x} \\ & & 1 \end{pmatrix} = \begin{pmatrix} a & ax & ay \\ & b & -b\overline{x} \\ & & \overline{a}^{-1} \end{pmatrix},$$

where $a \in E^\times$, $b \in E^1$ and $x, y \in E$ such that $y + \overline{y} + x\overline{x} = 0$. It is easy to observe that $a \in (1+\mathfrak{p}_E^{n-i}) \cap \mathfrak{o}_E^\times$ and $b \in E^1_{2i-n}$ if $g \in \gamma_i K_n \gamma_i^{-1}$. This implies that μ is trivial on $B \cap \gamma_i K_n \gamma_i^{-1}$ if $c(\mu_2) \leq 2i - n$ and $c(\mu_1) \leq n - i$.

Suppose that μ is trivial on $B \cap \gamma_i K_n \gamma_i^{-1}$. For all $a \in (1+\mathfrak{p}_E^{n-i}) \cap \mathfrak{o}_E^\times$, the element $t(a)$ belongs to $B \cap \gamma_i K_n \gamma_i^{-1}$. So we get $c(\mu_1) \leq n - i$. We claim that for each b in E^1_{2i-n}, there exists an element g in $B \cap \gamma_i K_n \gamma_i^{-1}$ whose $(2,2)$-entry is b. Then we get $\mu_2(b) = \mu(g) = 1$ because we have seen that the $(1,1)$-entry of $g \in B \cap \gamma_i K_n \gamma_i^{-1}$ lies in $(1+\mathfrak{p}_E^{n-i}) \cap \mathfrak{o}_E^\times$ and $c(\mu_1) \leq n - i$. So we obtain $c(\mu_2) \leq 2i - n$, which completes the proof.

We shall show the claim. Set

$$\mathcal{K} = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{o}_E \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{o}_E \end{pmatrix}.$$

We define an involution σ on $M_3(E)$ by $\sigma(X) = J^t\overline{X}J$, for $X \in M_3(E)$. Then we have $G = \{g \in \mathrm{GL}_3(E) \mid g\sigma(g) = 1\}$ and $K_n = (1+\mathcal{K}) \cap G$. For an element a in $\mathfrak{p}_F^{2i-n}$, we set

$$X = \begin{pmatrix} 0 & \varpi^{-i} a\sqrt{\epsilon} & \varpi^{-2i} a\sqrt{\epsilon} \\ 0 & a\sqrt{\epsilon} & \varpi^{-i} a\sqrt{\epsilon} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then one can observe that $1 - X$ lies in $\gamma_i(1+\mathcal{K})^\times \gamma_i^{-1}$. Since the group $\gamma_i(1+\mathcal{K})^\times\gamma_i^{-1}$ is σ-stable, the element $1 + X = \sigma(1-X)$ belongs to $\gamma_i(1+\mathcal{K})^\times\gamma_i^{-1}$. We put $g = (1-X)(1+X)^{-1}$. Then we see that g is an element in $B \cap \gamma_i K_n \gamma_i^{-1}$ whose $(2,2)$-entry is $(1 - a\sqrt{\epsilon})(1 + a\sqrt{\epsilon})^{-1}$. Suppose that $2i - n > 0$. Then [12] Theorem 2.13 (c) says that every element in E^1_{2i-n} has the form $(1 - a\sqrt{\epsilon})(1 + a\sqrt{\epsilon})^{-1}$, for some $a \in \mathfrak{p}_F^{2i-n}$. This completes the proof of the claim for $2i - n > 0$. Suppose that $2i - n = 0$. Then we have showed that the subgroup of E^1 consisting of the $(2,2)$-entries of elements in $B \cap \gamma_i K_n \gamma_i^{-1}$ contains $\{(1 - a\sqrt{\epsilon})(1 + a\sqrt{\epsilon})^{-1} \mid a \in \mathfrak{o}_F\}$. Now the claim follows from Lemma 1.1. □

Here we introduce functions in $\pi = \mathrm{Ind}_B^G \mu$, which form a basis for $V(n)$.

Definition 2.3. For $\lceil \frac{n}{2} \rceil \leq i \leq n$, we denote by $f_{n,i}$ the function in $V(n)$ which satisfies $f_{n,i}(\gamma_i) = 1$ and $\mathrm{supp}\, f_{n,i} = B\gamma_i K_n$. By Lemma 2.2, $f_{n,i}$ is well-defined if and only if i satisfies $c(\mu_2) \leq 2i - n$ and $c(\mu_1) \leq n - i$.

We shall determine the conductors of the parabolically induced representations of G, and give the dimension formula for the spaces of their oldforms.

Theorem 2.4. *Let π be a parabolically induced representation $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ of G, where μ_1 is a quasi-character of $E^\times$ and μ_2 is a character of E^1. Then*

(i) $N_\pi = 2c(\mu_1) + c(\mu_2)$;

(ii) For $n \geq N_\pi$, the functions $f_{n,i}$, $\frac{n + c(\mu_2)}{2} \leq i \leq n - c(\mu_1)$ constitute a basis for $V(n)$. In particular,

$$\dim V(n) = \left\lfloor \frac{n - N_\pi}{2} \right\rfloor + 1.$$

Proof. The theorem follows from Lemmas 2.1 and 2.2. □

The representation $\pi = \mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ admits the central character ω_π, which is given by

$$\omega_\pi(\iota(b)) = \mu_1(b)\mu_2(b), \ b \in E^1. \tag{2.5}$$

Proposition 2.6. *Let $\pi = \mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ be a parabolically induced representation of G. Then $N_\pi = n_\pi$ if and only if $c(\mu_1) = 0$. If this is the case, then we have $N_\pi = n_\pi = c(\mu_2)$.*

Proof. By (2.5), we have $n_\pi \le \max\{c(\mu_1), c(\mu_2)\}$. Suppose that $N_\pi = n_\pi$. Then by Theorem 2.4 (i), we obtain $\max\{c(\mu_1), c(\mu_2)\} = 2c(\mu_1) + c(\mu_2)$, whence $c(\mu_1) = 0$. Conversely, suppose that $c(\mu_1) = 0$. Then we have $\omega_\pi(\iota(b)) = \mu_2(b)$, for $b \in E^1$. So we get $n_\pi = c(\mu_2) = N_\pi$ by Theorem 2.4 (i). □

3. Reducible case

Every irreducible non-supercuspidal representation of G is a subquotient of $\mathrm{Ind}_B^G \mu$, for some quasi-character μ of T. In this section, we determine the conductors of the irreducible subquotients of $\mathrm{Ind}_B^G \mu$ in the case when $\mathrm{Ind}_B^G \mu$ is reducible.

3.1. Reducibility of parabolically induced representations.

Suppose that a parabolically induced representation $\pi = \mathrm{Ind}_B^G \mu_1 \otimes \mu_2$ is reducible. Due to [7], the reducibility of π is determined by the quasi-character $\widetilde{\mu}_1$ of $E^\times$ which is given by

$$\widetilde{\mu}_1(a) = \mu_1(a)\mu_2(\overline{a}/a), \ a \in E^\times.$$

There exist the following three reducible cases:

(R1) $\widetilde{\mu}_1 = |\cdot|_E^{\pm}$;

(R2) $\widetilde{\mu}_1|_{F^\times} = \omega_{E/F}|\cdot|_F^{\pm}$, where $\omega_{E/F}$ is the non-trivial character of $F^\times$ which is trivial on $N_{E/F}(E^\times)$;

(R3) $\widetilde{\mu}_1|_{F^\times} = 1$ and $\widetilde{\mu}_1 \ne 1$.

In all cases, the length of π is two. We denote by π_1 the unique irreducible generic subquotient of π and by π_2 the remaining one. By Theorem 1.3 (i), π_1 admits a newform. For $i = 1, 2$, we write V_i for the space of π_i. By [2] (2.4), we obtain

$$\dim V(n) = \dim V_1(n) + \dim V_2(n), \ n \ge 0. \tag{3.1}$$

So we have $N_{\pi_1} \ge N_\pi$. If π_2 also admits a newform, then we get $N_{\pi_2} \ge N_\pi$. The representations π_1 and π_2 have the same central character ω_π, and hence we have $n_\pi = n_{\pi_1} = n_{\pi_2}$.

3.2. Ramified case.

We shall consider the case when μ_1 is ramified. By Theorem 2.4 (i) and Proposition 2.6, we have $N_\pi = 2c(\mu_1) + c(\mu_2) \ge 2$ and $N_\pi > n_\pi$.

Proposition 3.2. *Suppose that μ_1 is ramified. Then*

(i) $\dim V_1(n) = \dim V(n)$ for all $n \ge 0$. In particular, $N_{\pi_1} = N_\pi$;

(ii) $V_2(n) = \{0\}$ for all $n \ge 0$.

Proof. Suppose that π_2 admits a newform. Then we have $N_{\pi_2} \ge N_\pi \ge 2$ and $N_{\pi_2} \ge N_\pi > n_\pi = n_{\pi_2}$. This contradicts [11] Theorem 4.4. So we conclude that π_2 has no K_n-fixed vectors. This implies (ii). The assertion (i) follows from (ii) and (3.1). □

3.3. Unramified case.

The case when μ_1 is unramified is slightly complicated. In this case, we have $N_\pi = n_\pi = c(\mu_2)$ by Proposition 2.6. Note that $\widetilde{\mu}_1$ agrees with μ_1 on $F^\times$. It is easy to observe that the conditions (R1)-(R3) are equivalent to the followings when μ_1 is unramified:

(RU1) $\mu_1 = |\cdot|_E^{\pm}$ and μ_2 is trivial;

(RU2) $\mu_1|_{F^\times} = \omega_{E/F}|\cdot|_F^{\pm}$;

(RU3) μ_1 is trivial and μ_2 is not trivial.

The injectivity of θ' is a main tool in our investigation. We determine the representations $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$ for which θ' is injective on $V(N_\pi)$.

Lemma 3.3. *Let $\pi = \mathrm{Ind}_B^G \mu_1 \otimes \mu_2$ be a parabolically induced representation, where μ_1 is an unramified quasi-character of $E^\times$. Then the level raising operator $\theta' : V(N_\pi) \to V(N_\pi + 1)$ is injective if and only if $\mu_1|_{F^\times} \ne \omega_{E/F}|\cdot|_F^{-1}$.*

Proof. By Theorem 2.4, we see that $N_\pi = c(\mu_2)$ and the function f_{N_π,N_π} forms a basis for the one-dimensional space $V(N_\pi)$. Thus, every function in $V(N_\pi)$ is supported on BK_{N_π}. This implies that an element f in $V(N_\pi)$ is not zero if and only if $f(1) \neq 0$. Similarly, all elements in $V(N_\pi + 1)$ are supported on $BK_{N_\pi+1}$. Hence $\theta' f \neq 0$ if and only if $\theta' f(1) \neq 0$. By (1.5), we have

$$\begin{aligned} \theta' f(1) &= f(\zeta^{-1}) + \sum_{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n}} f(u(0, x\sqrt{\epsilon})) \\ &= |\varpi^{-1}|_E \mu_1(\varpi^{-1}) f(1) + q f(1) \\ &= q(q\mu_1(\varpi^{-1}) + 1) f(1). \end{aligned}$$

So θ' is injective if and only if $\mu_1(\varpi) \neq -q$. Because we are assuming that μ_1 is unramified, the condition $\mu_1(\varpi) \neq -q$ is equivalent to $\mu_1|_{F^\times} \neq \omega_{E/F}|\cdot|_F^{-1}$, so this completes the proof. □

Firstly, we consider the case (RU1). Suppose that $\mu_1 = |\cdot|_E^{\pm}$ and μ_2 is trivial. Then we have $N_\pi = 0$ by Theorem 2.4 (i). It is well-known that π_1 is the Steinberg representation of G and π_2 is the trivial representation of G.

Proposition 3.4. *Suppose that μ_1 and μ_2 satisfy the condition (RU1). Then*

(i) $N_{\pi_1} = 2$ and

$$\dim V_1(n) = \left\lfloor \frac{n - N_{\pi_1}}{2} \right\rfloor + 1, \ n \geq N_{\pi_1};$$

(ii) $N_{\pi_2} = 0$ and

$$\dim V_2(n) = 1 \ \textit{for all} \ n \geq 0.$$

Proof. Since π_2 is the trivial representation of G, we have $\dim V_2(n) = 1$ for all $n \geq 0$. This implies (ii). For $n \geq 0$, we get $\dim V_1(n) = \dim V(n) - 1 = \lfloor \frac{n-2}{2} \rfloor + 1$ by (3.1) and Theorem 2.4 (ii). In particular, we obtain $N_{\pi_1} = 2$. This completes the proof of (i). □

Secondly, we shall consider the case (RU2). For the moment, we fix our notation as follows. Let μ_1 be the unramified quasi-character of $E^\times$ such that $\mu_1|_{F^\times} = \omega_{E/F}|\cdot|_F^{-1}$ and let μ_2 be a character of E^1. We set $\pi = \mathrm{Ind}_B^G \mu$, where $\mu = \mu_1 \otimes \mu_2$. By Lemma 3.3, the level raising operator θ' is zero on $V(N_\pi)$ since $\dim V(N_\pi) = 1$. It is known that (π, V) has the unique subrepresentation, which we denote by (τ, U). The quotient representation $\rho = \pi/\tau$ on $W = V/U$ is the unique subrepresentation of $\pi^w = \mathrm{Ind}_B^G \mu^w$, where $\mu^w = \overline{\mu_1}^{-1} \otimes \mu_2$. We write the space of π^w as V^w. Due to Theorem 2.4 (i), we have $N_\pi = N_{\pi^w} = c(\mu_2)$.

Lemma 3.5. *With the notation as above, we have*

$$\dim U(N_\pi) = \dim W(N_\pi + 1) = 1, \ \dim U(N_\pi + 1) = \dim W(N_\pi) = 0.$$

In particular, both τ and ρ admit newforms and those conductors are given by $N_\tau = N_\pi$, $N_\rho = N_\pi + 1$.

Proof. By Theorem 2.4 (ii), we have $\dim V(N_\pi) = \dim V(N_\pi + 1) = 1$. By [2] (2.4), for $n \geq 0$, we get an exact sequence

$$0 \to U(n) \to V(n) \to W(n) \to 0.$$

Under the identification $W = V/U$, the factor map $\theta' : V(N_\pi)/U(N_\pi) \to V(N_\pi+1)/U(N_\pi+1)$ coincides with $\theta' : W(N_\pi) \to W(N_\pi + 1)$. So θ' is zero on $W(N_\pi)$ because θ' on $V(N_\pi)$ is zero. Similarly, the factor map $\theta' : V^w(N_\pi)/W(N_\pi) \to V^w(N_\pi+1)/W(N_\pi+1)$ coincides with $\theta' : U(N_\pi) \to U(N_\pi + 1)$. By Lemma 3.3, the map $\theta' : V^w(N_\pi) \to V^w(N_\pi + 1)$ is bijective since $\dim V^w(N_\pi) = \dim V^w(N_\pi + 1) = 1$. So the restriction of θ' to $W(N_\pi)$ is injective and the factor map $\theta' : U(N_\pi) \to U(N_\pi + 1)$ is surjective. Because θ' on $W(N_\pi)$ is injective and zero, we have $\dim W(N_\pi) = 0$, and hence $\dim U(N_\pi) = \dim V(N_\pi) - \dim W(N_\pi) = 1$. Comparing

dimensions, we obtain $U(N_\pi) = V(N_\pi)$. Since $\theta' : U(N_\pi) \to U(N_\pi + 1)$ is surjective and zero, we have $\dim U(N_\pi + 1) = 0$, so that $\dim W(N_\pi + 1) = \dim V(N_\pi + 1) - \dim U(N_\pi + 1) = 1$. This completes the proof. □

Although the following lemma is not new, we give a proof for the reader's convenience.

Lemma 3.6. *With the notation as above, ρ is generic and τ is non-generic.*

Proof. Theorem 2.4 (ii) says that the space $V^w(N_\pi + 1)$ is one-dimensional. By Lemma 3.5, we have $W(N_\rho) = V^w(N_\pi + 1)$. It follows from Theorem 2.4 (ii) that every function in $V^w(N_\pi + 1)$ is supported on $BK_{N_\pi+1}$. Thus, for any non-zero element f in $V^w(N_\pi + 1)$, we have $f(1) \neq 0$. Applying the argument in the proof of Lemma 3.3, we get $\theta' f(1) \neq 0$. This implies that θ' on $W(N_\rho)$ is injective. So Lemma 1.7 (i) says that ρ must be generic. Since π has the unique irreducible generic subquotient, the remaining representation τ is not generic. □

Now we get the dimension formula of oldforms for representations in the case (RU2).

Proposition 3.7. *Suppose that μ_1 satisfies the condition (RU2). Then*
(i) $N_{\pi_1} = N_\pi + 1$ and

$$\dim V_1(n) = \left\lfloor \frac{n - N_{\pi_1}}{2} \right\rfloor + 1, \ n \geq N_{\pi_1};$$

(ii) $N_{\pi_2} = N_\pi$ and

$$\dim V_2(n) = \frac{1 + (-1)^{n - N_{\pi_2}}}{2}, \ n \geq N_{\pi_2}.$$

Proof. We may assume that μ_1 is the unramified quasi-character of $E^\times$ such that $\mu_1|_{F^\times} = \omega_{E/F}|\cdot|_F^{-1}$ since π and π^w have the same irreducible subquotients. Then Lemma 3.6 implies that $(\pi_1, V_1) = (\rho, W)$ and $(\pi_2, V_2) = (\tau, U)$. Due to Lemma 3.5, we have $N_{\pi_1} = N_\pi + 1$ and $N_{\pi_2} = N_\pi$.

As seen in the proof of Lemma 3.6, the operator θ' is injective on $V_1(N_{\pi_1})$. So by Lemma 1.7 (ii) and Proposition 1.6, we obtain $\dim V_1(n) = \left\lfloor \frac{n - N_{\pi_1}}{2} \right\rfloor + 1$, for $n \geq N_{\pi_1}$. The dimension formula of $V_2(n)$ follows from Theorem 2.4 (ii) and (3.1). □

Thirdly, we consider the case (RU3). Suppose that μ_1 is trivial and and μ_2 is not trivial. Then we have $N_\pi = c(\mu_2) \geq 1$ by Theorem 2.4 (i). In this case, both π_1 and π_2 are subrepresentation of π since μ is stable under the action of the long element of the affine Weyl group of G.

Proposition 3.8. *Suppose that μ_1 and μ_2 satisfy the condition (RU3). Then*
(i) $N_{\pi_1} = N_\pi$ and

$$\dim V_1(n) = \left\lfloor \frac{n - N_{\pi_1}}{2} \right\rfloor + 1, \ n \geq N_{\pi_1};$$

(ii) $\dim V_2(n) = 0$ *for all* $n \geq 0$.

Proof. There exists unique $i \in \{1, 2\}$ such that $N_{\pi_i} = N_\pi$ since $V(N_\pi)$ is one-dimensional. By Lemma 3.3, the operator θ' is injective on $V_i(N_{\pi_i}) = V(N_\pi)$. Thus Lemma 1.7 (i) implies that π_i must be generic. Hence we get $i = 1$. Due to Lemma 1.7 (ii) and Proposition 1.6, we obtain $\dim V_1(n) = \left\lfloor \frac{n - N_{\pi_1}}{2} \right\rfloor + 1$. Now Theorem 2.4 (ii) implies that $V_1(n) = V(n)$ for all $n \geq 0$, and hence we get $V_2(n) = \{0\}$ for all $n \geq 0$ by (3.1). □

4. Explicit newforms

In this section, we determine newforms for generic non-supercuspidal representations explicitly. Every irreducible non-supercuspidal representation π of G can be embedded into $\mathrm{Ind}_B^G\mu$, for some quasi-character μ of T. We shall realize the newforms for π as functions in $\mathrm{Ind}_B^G\mu$. This problem is easy unless π is the Steinberg representation of G:

Proposition 4.1. *Let (π, V) be an irreducible generic non-supercuspidal representation of G, which is not isomorphic to the Steinberg representation of G. Let μ_1 be a quasi-character of $E^\times$ and μ_2 a character of E^1 such that π is a subrepresentation of $\mathrm{Ind}_B^G\mu_1 \otimes \mu_2$. Then the function $f_{N_\pi, N_\pi - c(\mu_1)}$ is a newform for π. Here $f_{N_\pi, N_\pi - c(\mu_1)}$ is the function in Definition 2.3.*

Proof. By Theorem 2.4 (i), $\mathrm{Ind}_B^G\mu_1 \otimes \mu_2$ has conductor $2c(\mu_1) + c(\mu_2)$. If $\pi = \mathrm{Ind}_B^G\mu_1 \otimes \mu_2$, then the assertion follows from Theorem 2.4 (ii). Suppose that π is a proper submodule of $\mathrm{Ind}_B^G\mu_1 \otimes \mu_2$. Since π is not the representation in the case (RU1), it follows from Propositions 3.2, 3.7 and 3.8 that N_π equals to $2c(\mu_1) + c(\mu_2)$ or $2c(\mu_1) + c(\mu_2) + 1$. Theorem 2.4 (ii) says that the space of K_{N_π}-fixed vectors in $\mathrm{Ind}_B^G\mu_1 \otimes \mu_2$ is one-dimensional. So $V(N_\pi)$ is just the space of K_{N_π}-fixed functions in $\mathrm{Ind}_B^G\mu_1 \otimes \mu_2$. Thus the proposition follows from Theorem 2.4 (ii). $\square$

We shall determine newforms for the Steinberg representation St_G of G. Let π_1 be the Steinberg representation of G. Then π_1 is the unique subrepresentation of $\pi = \mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$, where $\mu_1 = |\cdot|_E$ and $\mu_2 = 1$. We have $N_{\pi_1} = 2$, $N_\pi = 0$ and $n_{\pi_1} = n_\pi = 0$ by Propositions 2.6, 3.4 (i) and Theorem 2.4 (i). We write V and V_1 for the spaces of π and π_1 respectively. Due to Theorem 2.4 (ii), $V(2)$ is the two-dimensional subspace of V spanned by $f_{2,1}$ and $f_{2,2}$. The space $V_1(2)$ of newforms for π_1 is a one-dimensional subspace of $V(2)$.

Proposition 4.2. *With the notation as above, a function f in $V(2)$ lies in $V_1(2)$ if and only if $f(1) = -q(q-1)f(\gamma_1)$. In particular, $q(q-1)f_{2,2} - f_{2,1}$ is a newform for the Steinberg representation of G.*

Proof. Since π has trivial central character, the group Z_1K_2 acts on $V(2)$ trivially. We define a level lowering operator $\delta : V(2) \to V(1)$ by

$$\delta v = \frac{1}{\mathrm{vol}(K_1 \cap (Z_1K_2))} \int_{K_1} \pi(k)v dv, \ v \in V(2).$$

The space $V_1(2)$ is contained in $\ker\delta$ since $V_1(1) = \{0\}$. We shall show that a function f in $V(2)$ lies in the kernel of δ if and only if $f(1) = -q(q-1)f(\gamma_1)$. Then $\ker\delta$ is of dimension one, and hence coincides with $V_1(2)$.

It follows from [10] Lemma 4.9 that δ has the following form:

$$\delta v = \sum_{\substack{y \in \mathfrak{p}_E/\mathfrak{p}_E^2 \\ z \in \mathfrak{p}_F/\mathfrak{p}_F^2}} \pi(\hat{u}(y, z\sqrt{\epsilon} - y\overline{y}/2))v + \sum_{y \in \mathfrak{p}_E^{-1}/\mathfrak{o}_E} \pi(\zeta u(y, -y\overline{y}/2))v, \ v \in V(2).$$

For any element f in $V(2)$, we have

$$\begin{aligned}
\delta f(1) &= \sum_{\substack{y\in\mathfrak{p}_E/\mathfrak{p}_E^2\\ z\in\mathfrak{p}_F/\mathfrak{p}_F^2}} \pi(\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2))f(1) + \sum_{y\in\mathfrak{p}_E^{-1}/\mathfrak{o}_E} \pi(\zeta u(y,-y\overline{y}/2))f(1)\\
&= \sum_{\substack{y\in\mathfrak{p}_E/\mathfrak{p}_E^2\\ z\in\mathfrak{p}_F/\mathfrak{p}_F^2}} f(\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2)) + \sum_{y\in\mathfrak{p}_E^{-1}/\mathfrak{o}_E} f(\zeta u(y,-y\overline{y}/2))\\
&= \sum_{\substack{y\in\mathfrak{p}_E/\mathfrak{p}_E^2\\ z\in\mathfrak{p}_F/\mathfrak{p}_F^2}} f(\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2)) + q^{-2}\mu_1(\varpi)\sum_{y\in\mathfrak{p}_E^{-1}/\mathfrak{o}_E} f(u(y,-y\overline{y}/2))\\
&= \sum_{\substack{y\in\mathfrak{p}_E/\mathfrak{p}_E^2\\ z\in\mathfrak{p}_F/\mathfrak{p}_F^2}} f(\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2)) + \mu_1(\varpi)f(1).
\end{aligned}$$

We shall compute $f(\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2))$, for $y\in\mathfrak{p}_E/\mathfrak{p}_E^2$ and $z\in\mathfrak{p}_F/\mathfrak{p}_F^2$.

(i) Suppose that $y\in\mathfrak{p}_E^2$ and $z\in\mathfrak{p}_E^2$. Then we have $f(\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2)) = f(1)$ since $\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2)$ lies in K_2.

(ii) If $y\notin\mathfrak{p}_E^2$ and $z\in\mathfrak{p}_E^2$, then $\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2) = \hat{u}(y,-y\overline{y}/2)\hat{u}(0, z\sqrt{\epsilon}) \equiv \hat{u}(y,-y\overline{y}/2)$ (mod K_2). There exists $a\in\mathfrak{o}_E^\times$ such that $t(a)\hat{u}(y,-y\overline{y}/2)t(a)^{-1} = \hat{u}(\varpi,-\varpi^2/2) = \gamma_1$. So we have $f(\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2)) = f(\gamma_1)$.

(iii) Suppose that $z\notin\mathfrak{p}_E^2$. Then $x = z\sqrt{\epsilon}-y\overline{y}/2$ lies in $\mathfrak{p}_E\backslash\mathfrak{p}_E^2$. We have

$$\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2) = u(-\overline{y}/\overline{x}, 1/x)\mathrm{diag}(\varpi^2/\overline{x}, -\overline{x}/x, \varpi^{-2}x)t_2u(-\overline{y}/x, 1/\overline{x}).$$

Since $t_2u(-\overline{y}/x, 1/\overline{x})\in K_2$ and $\varpi^2/\overline{x}\in\varpi\mathfrak{o}_E^\times$, we obtain

$$f(\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2)) = q^{-2}\mu_1(\varpi)\mu_2(-\overline{x}/x)f(1) = q^{-2}\mu_1(\varpi)f(1).$$

Therefore we get

$$\begin{aligned}
\delta f(1) &= \sum_{\substack{y\in\mathfrak{p}_E/\mathfrak{p}_E^2\\ z\in\mathfrak{p}_F/\mathfrak{p}_F^2}} f(\hat{u}(y, z\sqrt{\epsilon}-y\overline{y}/2)) + \mu_1(\varpi)f(1)\\
&= f(1) + (q^2-1)f(\gamma_1) + q^2(q-1)q^{-2}\mu_1(\varpi)f(1) + \mu_1(\varpi)f(1)\\
&= (q\mu_1(\varpi)+1)f(1) + (q^2-1)f(\gamma_1)\\
&= (q^{-1}+1)f(1) + (q^2-1)f(\gamma_1).
\end{aligned}$$

By Lemma 2.1, we have $G = BK_1$. Therefore $\delta f\in V(1)$ is zero if and only if $\delta f(1) = 0$. So we conclude that $\delta f = 0$ if and only if $f(1) = -q(q-1)f(\gamma_1)$. This completes the proof. □

Corollary 4.3. *Suppose that an irreducible generic representation (π, V) of G is a subrepresentation of $\mathrm{Ind}_B^G\mu_1\otimes\mu_2$, where μ_1 is an unramified quasi-character of $E^\times$. We regard elements in V as functions in $\mathrm{Ind}_B^G\mu_1\otimes\mu_2$. Then any non-zero element in $f\in V(N_\pi)$ satisfies $f(1)\neq 0$.*

Proof. If π is not isomorphic to the Steinberg representation of G, then it follows from Proposition 4.1 that a non-zero element in $f\in V(N_\pi)$ is a non-zero multiple of f_{N_π,N_π}. The assertion follows because the support of f_{N_π,N_π} is BK_{N_π}. Suppose that π is the Steinberg representation of G. Then we have $\mu_1 = |\cdot|_E$ and $\mu_2 = 1$. By Proposition 4.2, a non-zero element in $f\in V(N_\pi)$ is a non-zero multiple of $q(q-1)f_{2,2} - f_{2,1}$. Since $(q(q-1)f_{2,2} - f_{2,1})(1) = q(q-1)$, the proof is complete. □

5. Test vectors for the Whittaker functional

We close this paper by showing that newforms for generic representations of G are test vectors for the Whittaker functional.

Proposition 5.1. *Let (π, V) be an irreducible generic representation of G. Then we have $W_v(1) \neq 0$ for all non-zero $v \in V(N_\pi)$.*

Proof. The assertion follows from [11] Theorem 4.12 if π satisfies $N_\pi \geq 2$ and $N_\pi > n_\pi$. By [11] Corollary 5.5, all generic supercuspidal representations satisfy this condition. So we may assume that π is not supercuspidal. As usual, we embed π into $\tau = \mathrm{Ind}_B^G \mu_1 \otimes \mu_2$. If μ_1 is ramified, we have $N_\pi \geq N_\tau = 2c(\mu_1) + c(\mu_2) \geq 2$ and $N_\pi \geq N_\tau > n_\tau = n_\pi$ by Theorem 2.4 (i) and Proposition 2.6. So we may suppose μ_1 is unramified.

(I) Suppose that $N_\pi = 0$. We claim that τ is irreducible. Then $\pi = \mathrm{Ind}_B^G \mu_1 \otimes \mu_2$ is an unramified principal series representation by Theorem 2.4 (i). Hence the assertion follows from [4] Theorem 5.4. We shall show the claim. We assume that τ is reducible. If μ_1 and μ_2 satisfy the conditions (RU1) or (RU2), then by Propositions 3.4 and 3.7, N_π must be positive. Suppose that μ_1 and μ_2 satisfy the condition (RU3). Then we have $N_\pi \geq N_\tau = c(\mu_2) > 0$ by Theorem 2.4 (i). This completes the proof of the claim.

(II) Suppose that $N_\pi \geq 1$. Then, by Lemma 1.7, it is enough to show that θ' is injective on $V(N_\pi)$. We regard any element in V as a function in $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$. By Corollary 4.3, any non-zero element in $V(N_\pi)$ satisfies $f(1) \neq 0$. Since π is the generic subrepresentation of $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$, Lemma 3.6 implies $\mu_1|_{F^\times} \neq \omega_{E/F} |\cdot|_F^{-1}$. Applying the argument in the proof of Lemma 3.3, we obtain $\theta' f(1) \neq 0$. This implies that θ' is injective. The proof is now complete. $\square$

By Proposition 1.6, we obtain the following

Corollary 5.2. *Let (π, V) be an irreducible generic representation of G. Then, for $n \geq N_\pi$, the set $\{\theta'^i \eta^j v \mid i + 2j + N_\pi = n\}$ forms a basis for $V(n)$. In particular,*

$$\dim V(n) = \left\lfloor \frac{n - N_\pi}{2} \right\rfloor + 1.$$

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